Link Theory –

from Computation to Quantum Physics

or

How to build a Universe using only things
found lying around the math department

Part II: the Miracle

Thomas Etter and Richard Shoup
Boundary Institute
Los Altos, California
etter@boundaryinstitute.com
shoup@boundaryinstitute.com

Abstract: Part I of this paper presents the basic ideas and mechanics of Link Theory, along with several examples from the realm of digital circuits, computation, and physics. In Part II, we show how the same principles can be used to derive the core laws of Quantum Physics, and how these laws are properties of the mathematics, not miracles, and not in any way derived from properties of physical matter.

0. INTRODUCTION

In the following sections, we discuss matrix and Link Table representations for states, projections, and transformations in quantum mechanics. We shall see how Link Theory leads to the core laws of quantum mechanics, and more important, to a generalization of these core laws that encompasses classical and other structures as well. By the core laws, we mean von Neumann's generalized statements of the Born probability rule, the average quantity rule and the Schroedinger equation. These three laws by no means exhaust quantum mechanics, since they make no mention of space, time, energy or any other specifically physical quantity. They do, however, encompass the essential novelty and strangeness of quantum mechanics, and are precisely that part of physics needed for the logical design of quantum computers.

The core laws will first be introduced without any Link Theory. Next, we will see how to derive them as general theorems of Link Theory. Finally, we look at quantum measurement or *quantization*, and show that it *does not require any miracle or "collapse" of any wave function at all*. Our treatment will necessarily be brief and informal; a fuller and more rigorous account can be found in (Etter and Noyes, 1998).

Somewhere in the distant past, arithmetic began as accounting. The need to tally cattle and other commodities led gradually to the concept of number and operations of addition, subtraction, and so on. Eventually, the idea of number and the calculating methods of arithmetic became abstract, separate from their applications. Similarly, Gaussian distributions were first derived from various physical situations, and only later abstracted into mathematics. Until Newton, many people considered the directions "up" and "down" to be immutable properties of space itself.

Today, our theories of quantum phenomena includes several new forms of mathematics which are considered, at least implicitly, to be properties of matter, belonging strictly to the domain of physics. Typically, the Quantum Core laws discussed below are introduced in a complicated way as *Laws of Physics* only at an advanced level of study.

It is the purpose of this work to show that another abstraction is possible: The Quantum Core turns out to be an unavoidable consequence of certain simple mathematical definitions in Link Theory, which is a general theory of abstract structure based on Russell and Whitehead's theory of relations (See Part I of this report for a discussion of the basics of Link Theory.)

In Step One, we will derive the "classical" case of the quantum core laws without using either Link Theory or physics, only common sense. Next, in Step Two we introduce the "miracle" needed in standard quantum theory to get the core laws in their full generality. Finally, we show how this miracle can be replaced by straightforward combinatorial analysis, *thereby factoring the core laws entirely out of physics.*

1. STEP ONE

Below is a statement of the core laws of quantum mechanics. **P**, **Q**, **D**, and **T** are matrices. We will see that these laws are just *common sense* (and contain no physics) for the special case when **P**, **Q**, and **D** commute. (We can think of this special case as describing the classical limit of quantum behavior.)

The probability rule: $\text{Prob}(\mathbf{P},\mathbf{D}) = \text{trace}(\mathbf{PD})$

The average rule: $\text{Average}(\mathbf{Q},\mathbf{D}) = \text{trace}(\mathbf{QD})$

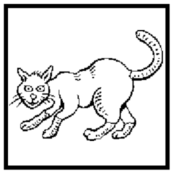
The state transformation rule: $\mathbf{D}' = \mathbf{T}^{-1}\mathbf{D}\mathbf{T}$

Let us start with a set of objects classified by *states*, some *statements* about them, and numerical functions of these states called *quantities*.

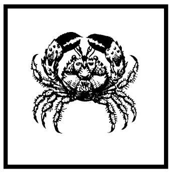
1.1 Classical states and statements

Suppose there are 100 children's blocks on the table – 60 birds, 10 cats, and 30 crabs.

60 birds



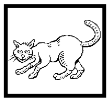
10 cats

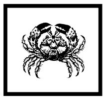


30 crabs

The *state* of a block, i.e. Bird, Cat, or Crab, will be associated with a 3x3 matrix of 0's and 1's called the *state matrix* **S**. (The appropriateness of this representation will become clear later.) Each matrix contains a single 1 in the diagonal thus:

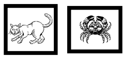
$$\begin{array}{ccc} \text{1,1 } \textit{Bird} & \text{2,2 } \textit{Cat} & \text{3,3 } \textit{Crab} \\ \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} \end{array}$$





Any *statement* about block X will also be assigned a matrix **P** called a *projection* that shows the states for which it is true. Here are some examples:

$$\begin{array}{cccc} \text{X has more than two legs} & \text{X lays eggs} & \text{X is a crab} & \text{X is anything} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} & \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} & \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} \end{array}$$



1.2. Boolean logic as matrix algebra

Now we can perform simple logical operations using this representation. For example, the matrix of the statement (A AND B) is just the matrix of A multiplied by the matrix of B:

$$\begin{array}{ccc} \text{X has more than two legs AND X lays eggs} & & \text{X is a crab} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} & = & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} \end{array}$$

The matrix of statement (NOT A) is the identity matrix minus the matrix of A:

$$\overset{\text{X is NOT a crab}}{\begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}} = \overset{\text{X is anything}}{\begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix}} - \overset{\text{X is a crab}}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix}}$$

1.3. Density matrices and probabilities

The *density matrix* **D** shows the proportion of creatures in each state. Since there are 30 birds, 10 cats and 60 crabs in this case, we have

$$\mathrm{D} = \begin{pmatrix} .3 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & .6 \end{pmatrix} .$$

1.4. The generalized Born probability rule

The *trace* of a matrix is defined as the sum of its diagonal elements, so

$$\mathrm{trace}(\mathbf{D}) = .3 + .1 + .6 = 1.0 .$$

The Born probability rule of quantum mechanics states that the probability of statement **P** about a collection having density matrix **D** is given by the trace of the product of **P** and **D**:

$$\mathrm{Prob}(\mathbf{P},\mathbf{D}) = \mathrm{trace}(\mathbf{PD}) .$$

Example:

$$\text{Prob(X is not a crab)} = \text{trace}\left(\begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} .3 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & .6 \end{pmatrix} \right) = .4$$

1.5. Quantities

A *quantity* is defined as any numerical function q(**S**) of state **S**. It will be represented by a diagonal matrix **Q** that has q(**S**) in each **S**'s diagonal place. Here is the matrix **Q** for the function "number of legs" :

$$\mathrm{Q} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} .$$

1.6. Averages

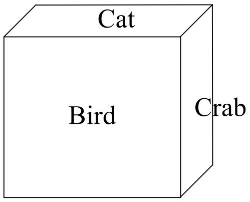
To find the *average* of a quantity in a collection, we multiply the quantity for each state by the proportion of members in that state and add the results. This is the same as multiplying the quantity matrix by the density matrix.

Definition: Average(**Q**,**D**) = trace(**QD**). So for this case,

$$\text{Average}(\mathbf{Q},\mathbf{D}) \;=\; \left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} .3 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & .6 \end{pmatrix} \right) \;=\; 5.8\,.$$

1.7. Transformations

So far we have been looking down on the blocks. Actually, all of the blocks are identical; one animal is on the left & right faces, another on the front & back faces, and the third on the top & bottom faces.



1.8. Permutations of matrices

In general, any linear transformation **T** of a matrix **M** can be written:

$$\mathbf{M'} \;=\; \mathbf{T}^{-1}\,\mathbf{M}\,\mathbf{T}$$

A *permutation* is a special case of a transformation, defined as a matrix with a single 1 in each row and each column, and 0's elsewhere. Here is the *cyclic* permutation **T** = [2, 3, 1] applied to **D**:

$$\begin{array}{ccccc} \mathbf{D'} & & \mathbf{T^{-1}} & \mathbf{D} & \mathbf{T} \\ \begin{pmatrix} .6 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & .1 \end{pmatrix} & = & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} .3 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & .6 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

In this example, the application of T has permuted birds to cats, cats to crabs, and crabs to birds.

1.9. The core laws

With just the above simple considerations, we have already seen in primitive form the three core laws of quantum mechanics, expressed as pure mathematics, without invoking any Physics at all:

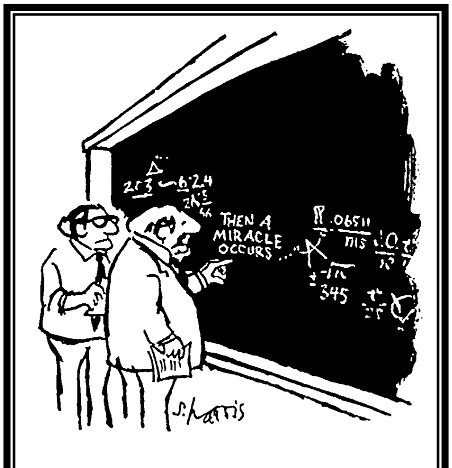
The Born Probability Rule: Prob(**P**,**D**) = trace(**PD**)

The Average Rule: Average(**Q**,**D**) = trace(**QD**)

The Transformation Rule for states: $\mathbf{D}' = \mathbf{T}^{-1}\mathbf{D}\mathbf{T}$

Now let's make the simple generalizations and interpretations which show how these laws fully represent quantum situations, dynamics, and measurement.

2. STEP TWO - THE MIRACLE OF QUANTIZATION



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

The Standard Miracle that produces the quantum core is straightforward given the preliminaries above**:** Allow the transformation **T** to include *all rotations*, not just cyclic permutations.

More formally, we generalize to *closure under rotation*: The class of transformation matrices **T** is enlarged from the class of permutations of axes to the class generated by all rotations and reflections. This type of matrix is also known as a *unitary* transformation – one whose inverse is equal to its transpose. That is, $\mathbf{T}^{-1} = \mathbf{T}^{*}$.

The core laws are assumed to still hold. This requires that the **P**, **Q**, and **D** classes be enlarged to include all unitary transforms of their diagonal matrices as well.

All quantum weirdness (interference, EPR non-locality, etc.) resides in the quantum core. Note that, when viewed this way, the quantum core says nothing about space, time or matter.

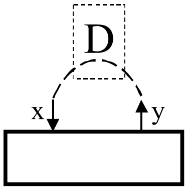
2.1. Step Two made explicit

Step 2 above makes no sense at all in terms of classical experience; It's a weird and seemingly ad-hoc intruder into the common-sense world. We'll now see how to completely represent it by simple case counting in Link Theory. The theory needed is only a slight extension of that described in Part I, in that we will now allow cases to count *negatively* as well as positively.

The crucial step is to *interpret density matrices as the matrices of broken links* (see Part I, section 1.5). This leads immediately to the Born Rule and Average Rule at every link, and to the Transformation Rule for the links on either side of any doubly-linked two-column table. The quantum form of these rules follows from the single additional assumption that every link is symmetrical in its two linked variables.

2.2. Breaking a link

Breaking a link between variables $\mathbf{x}$ and $\mathbf{y}$ defines a matrix which will be called the *link state* of $\mathbf{x}$ or of $\mathbf{y}$. Link states defined in this way act as density matrices, thus $\mathbf{D}$ is the density matrix which would result if we broke the link between $\mathbf{x}$ and $\mathbf{y}$:



2.3. Types of density matrix

Not all density matrices defined by breaking a link are *quantum states*. The defining condition for a quantum state is that the matrix is unchanged by reversing $\mathbf{x}$ and $\mathbf{y}$. A matrix with this symmetry property is called *self-adjoint.*

A state is *pure* if $\mathbf{x}$ and $\mathbf{y}$ are independent. If a pure state is self-adjoint, then it is a *quantum pure state*, and thus $\mathbf{x}$ and $\mathbf{y}$ are the same vector. The (normalized) counts in vector $\mathbf{x}$ (or $\mathbf{y}$) are what in physics are called the *amplitudes* of the quantum state.

In a *causal state* (which is usually pure), one vector is "white" (all counts equal), while the other vector carries all the information.

Quantum state: $\mathbf{D} = \mathbf{D}^*$.

Quantum pure state: $\mathbf{D} = |\,\mathbf{v} > < \mathbf{v}\,|$. The vector $\mathbf{v}$ is the wave function.

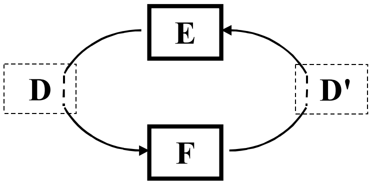
Causal pure state: $\mathbf{D} = |\,\mathbf{p} > < \mathbf{1}\,|$,
where $\mathbf{p}$ is a probability distribution vector and $\mathbf{1}$ is the white vector.

Note that quantum and causal are only two ideal limiting cases among an infinite variety of state types.

2.4. The Born Rule and Average Rule

Breaking a quantum link is *counterfactual*, i.e., we never *actually* encounter the broken ends $\mathbf{x}$ and $\mathbf{y}$. We always see $\mathbf{x}$ and $\mathbf{y}$ *linked*, as a single variable. Since their broken distributions are equal, the actual distribution on $\mathbf{x}$ (or $\mathbf{y}$) is always the square of that distribution; this proves the well-known version of Born's Rule which says that probability is the square of amplitude. From this, the von Neumann generalization of Born's Rule and the Average Law can be derived for diagonal operators by the same common-sense reasoning that we used in Step 1, and extended to non-diagonal operators by the generalized Schroedinger equation (see below) together with the theorem that trace(**AB**) = trace(**BA**).

2.5. Transformations of state

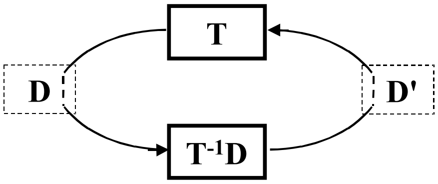


Suppose the tables **E** and **F** are linked as shown, with **D** and **D'** as link states on their respective linked variables. If the matrix of table **E** has an inverse, then $\mathbf{D'} = \mathbf{E}^{-1}\mathbf{D}\mathbf{E}$.

Proof: $\mathbf{D} = \mathbf{EF}$ (only **D** is broken), so $\mathbf{F} = \mathbf{E}^{-1}\mathbf{D}$. But $\mathbf{D'} = \mathbf{FE}$ (only **D'** is broken), so by substitution $\mathbf{D'} = \mathbf{E}^{-1}\mathbf{D}\mathbf{E}$.

This is the core transformation law. If **E** transforms any quantum state **D** into another quantum state **D'**, it can be shown that **E** must be unitary.

2.6. Example: cyclic permutation

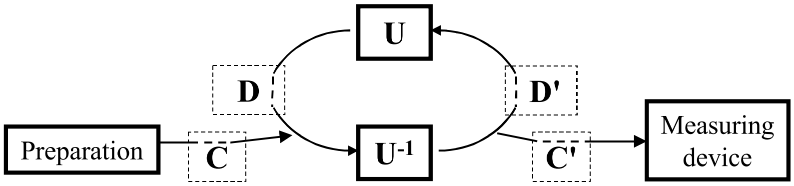


Let us return to the special case of permutation, where **E** is the permutation matrix **T** of section 1.8, and **F** is $\mathbf{T}^{-1}\mathbf{D}$, where **D** is the animal block density matrix of section 1.3. Then $\mathbf{D'} = \mathbf{T}^{-1}\mathbf{D}\mathbf{T}$.

Proof: The left link state is $\mathbf{T}\mathbf{T}^{-1}\mathbf{D} = \mathbf{D}$, while the right link state is $\mathbf{T}^{-1}\mathbf{D}\mathbf{T} = \mathbf{D'}$.

2.7. Quantum measurement

Now we are prepared to present the complete situation of quantum measurement.



U is any unitary transformation matrix. **C** is the (causal) density matrix of the *preparation*, a state vector which defines the initial state **D** of the quantum system to be measured. **C'** is the (causal) density matrix of the *measurement* which couples the final state **D'** to the measuring device. See (Etter and Noyes, 1998) for a more complete presentation.

There is no actual wave function to collapse -- it's counterfactual.

3. REFERENCES AND RELATED WORK

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